Tutorial 2: Selected problems of Assignment 2

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Assumption Throughout the tutorial, $f = f_1 + i f_2 : [-\pi, \pi] \to \mathbb{C}$ is a complex-valued 2π-periodic integrable function with Fourier series $S(f) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$

Q1) (Ex 2, Q4) Suppose in addition, f is differentiable such that $f' : [-\pi, \pi] \to \mathbb{C}$ satisfies the Assumption.

Show that $\widehat{f'}(n) = in\hat{f}(n)$, $\forall n \in \mathbb{Z}$

Sol) Method 1: Assuming integration by parts for complex-valued functions.

Fact For any such functions $g, h : [-\pi, \pi] \to \mathbb{C}$,

$$\int_{-\pi}^{\pi} g'(x)h(x)\,dx = [g(x)h(x)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} g(x)h'(x)\,dx$$

$$\therefore 2\pi\widehat{f'}(n) = \int_{-\pi}^{\pi} f'(x)e^{-inx}dx \overset{\text{Fact}}{=} [f(x)e^{-inx}]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(f(x)(-in\,e^{-inx}))dx = in\,2\pi\hat{f}(n)$$

Method 2: Using integration by parts for real-valued functions.

$$2\pi\widehat{f'}(n) = \int_{-\pi}^{\pi} f'(x)e^{-inx}dx = \int_{-\pi}^{\pi}(f_1'(x) + i f_2'(x))(\cos nx - i\sin nx)\,dx$$

$$= \int_{-\pi}^{\pi}(f_1'(x)\cos nx + f_2'(x)\sin nx)dx + i\int_{-\pi}^{\pi}(f_2'(x)\cos nx - f_1'(x)\sin nx)dx$$

$$= \left\{[f_1(x)\cos nx + f_2(x)\sin nx]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(f_1(x)(-n\sin nx) + f_2(x)(n\cos nx))dx\right\}$$

$$+ i\left\{[f_2(x)\cos nx - f_1(x)\sin nx]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(f_2(x)(-n\sin nx) - f_1(x)(n\cos nx))dx\right\}$$

$$= n\left(\int_{-\pi}^{\pi}(f_1(x)\sin nx - f_2(x)\cos nx)dx + i\int_{-\pi}^{\pi}(f_2(x)\sin nx + f_1(x)\cos nx)dx\right.$$

$$= in\int_{-\pi}^{\pi}(f_1(x) + i f_2(x))(\cos nx - i\sin nx)\,dx = in\,(2\pi\hat{f}(n))$$

Q2) [Ex 2, Q7] Define $F: [-\pi, \pi] \to \mathbb{C}$ by $F(x) = \int_{-\pi}^{x} f(t)\,dt$

(a) Show that F is 2π periodic $\iff \hat{f}(0) = 0$

(b) Suppose in addition, f is continuous with $\hat{f}(0) = 0$, show that

F satisfies the Assumption with $\hat{F}(n) = \frac{1}{in}\hat{f}(n)$, $\forall\, n \neq 0$

Sol) (a) Note that $F(x+2\pi) = \int_{-\pi}^{x+2\pi} f(t)\,dt = \int_{-\pi}^{x} f(t)\,dt + \int_{x}^{x+2\pi} f(t)\,dt$

$$= F(x) + \int_{-\pi}^{\pi} f(t)\,dt = F(x) + 2\pi\hat{f}(0)$$

$\therefore$ F is 2π-periodic $\iff F(x) = F(x+2\pi),\ \forall\, x \in [-\pi, \pi] \iff \hat{f}(0) = 0$

(b) Since f is continuous, by Fundamental Theorem of Calculus,

F is differentiable on $[-\pi, \pi]$ with $F'(x) = f(x)$ satisfying the Assumption.

Moreover, since $\hat{f}(0) = 0$, by (a), F also satisfies the Assumption.

$\therefore$ By Q1, $\hat{f}(n) = \widehat{F'}(n) = in\hat{F}(n)$, $\forall\, n \in \mathbb{Z}$

$\therefore$ $\forall\, n \neq 0$, $\hat{F}(n) = \frac{1}{in}\hat{f}(n)$.

Q3) (Ex 2, Q10) Suppose in addition, f is Lipschitz continuous.
Show that there exists $C>0$ such that for all $n \neq 0 \in \mathbb{Z}$,

$$|\hat{f}(n)| \leq \frac{C}{|n|}$$

Sol) Since f is Lipschitz continuous, $\exists\, L>0$ such that for all $x, y \in [-\pi, \pi]$,

$$|f(x)-f(y)| \leq L\,|x-y|.$$

Note that for all $n \neq 0$, $2\pi \hat{f}(n) = \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$

$$(\text{Put } x = y+\tfrac{\pi}{n}) \qquad = \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y+\tfrac{\pi}{n}) e^{-in(y+\frac{\pi}{n})}\,dy$$

$$= -\int_{-\pi}^{\pi} f(y+\tfrac{\pi}{n}) e^{-iny}\,dy$$

$$\therefore |\hat{f}(n)| = \frac{1}{2\pi}\left(\frac{1}{2}\left|\int_{-\pi}^{\pi} (f(x)-f(x+\tfrac{\pi}{n})) e^{-inx}\,dx\right|\right)$$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x)-f(x+\tfrac{\pi}{n})|\,|e^{-inx}|\,dx$$

$$\leq \frac{1}{4\pi}\cdot 2\pi \cdot L\frac{\pi}{|n|} = \frac{\pi L}{2|n|} = \frac{C}{|n|} \quad \text{by setting } C = \frac{\pi}{2}L$$